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Find all real solutions of the following system of equations

$x_k x_{k+1} + 1 = 4x_k, k = 1,2,\ldots,2019$

$x_{2020}x_1 + 1 = 4x_1$

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We will solve the following general version of the problem:

For any natural $m \geq 2$ find all real solutions of the following system of equations

(**1**) $\begin{cases} x_k x_{k+1} + 1 = 4x_k, k = 1,2,\ldots,m-1 \\ x_m x_1 + 1 = 4x_1 \end{cases}$.

Let $t_k := 4 - x_k, k = 1,2,\ldots,n.$ Then (**1**)$\Leftrightarrow \begin{cases} x_{k+1} = \dfrac{1}{4-x_k}, k = 1,2,\ldots,m-1 \\ x_1 = \dfrac{1}{4-x_m} \end{cases} \Leftrightarrow$

(**2**) $\begin{cases} t_{k+1} = 4 - \dfrac{1}{t_k}, k = 1,2,\ldots,m-1 \\ t_1 = 4 - \dfrac{1}{t_m} \end{cases}$ and $x_k = 4 - t_k, k = 1,2,\ldots,m.$

Consider a sequence $(t_n)_{n\in\mathbb{N}}$ defined by $t_1 = t \neq 0, t_{n+1} = 4 - \dfrac{1}{t_n}, n \in \mathbb{N}.$

Let $p_n := t_1 t_2 \ldots t_n, n \in \mathbb{N}$ and $p_0 = 1.$ Then $p_1 = t,\ t_n := \dfrac{p_n}{p_{n-1}}, n = 1,2,\ldots,$ and $t_{n+1} = 4 - \dfrac{1}{t_n}$ becomes $\dfrac{p_{n+1}}{p_n} = 4 - \dfrac{p_{n-1}}{p_n} \Leftrightarrow p_{n+1} - 4p_n + p_{n-1} = 0, n \in \mathbb{N}.$

Consider sequence (a_n) defined by $a_0 = 0, a_1 = 1$ and $a_{n+1} - 4a_n + a_{n-1} = 0, n \in \mathbb{N}.$

Since $\det\begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{n-1} \end{pmatrix} \neq 0, n \in \mathbb{N}$ ($a_{n+2}a_n - a_{n+1}^2 = a_{n+1}a_{n-1} - a_n^2, n \in \mathbb{N}$ implies $a_{n+1}a_{n-1} - a_n^2 = a_2 a_0 - a_1^2 = -1$) then p_n can be represent as linear combination of a_n and a_{n+1}, namely $p_n = c_1 a_n + c_2 a_{n+1}, n \in \mathbb{N} \cup \{0\}$ for some $c_1, c_2 \in \mathbb{R}$.

Using initial conditions $p_0 = 1, p_1 = t$ we obtain $\begin{cases} 1 = c_1 a_0 + c_2 a_1 \\ t = c_1 a_1 + c_2 a_2 \end{cases} \Leftrightarrow \begin{cases} c_2 = 1 \\ c_1 = t-4 \end{cases}$.

Hence, $p_n = (t-4)a_n + a_{n+1}$ and $t_n = \dfrac{(t-4)a_n + a_{n+1}}{(t-4)a_{n-1} + a_n}, n \in \mathbb{N}.$

Coming back to the system (**2**) and noting that* $a_n > 0$ for any $n \in \mathbb{N}$ we obtain

$t_1 = 4 - \dfrac{1}{t_m} \Leftrightarrow t = \dfrac{(t-4)a_m + a_{m+1}}{(t-4)a_{m-1} + a_m} \Leftrightarrow (t-4)a_m + a_{m+1} = t(t-4)a_{m-1} + ta_m \Leftrightarrow$

$a_{m-1}t^2 - 4a_{m-1}t + (4a_m - a_{m+1}) = 0 \Leftrightarrow a_{m-1}t^2 - 4a_{m-1}t + a_{m-1} = 0, m \geq 2 \Leftrightarrow$

$t^2 - 4t + 1 = 0 \Leftrightarrow t = 2 \pm \sqrt{3} \Leftrightarrow x = 4 - \left(2 \pm \sqrt{3}\right) = 2 \mp \sqrt{3}.$

Thus, $x_1 = x_2 = \ldots = x_m = 2 + \sqrt{3}$ and $x_1 = x_2 = \ldots = x_m = 2 - \sqrt{3}$ are all real solution of the system (**1**).

* Since $a_1 = 1, a_2 = 4$ then for any $n \in \mathbb{N}$ assuming $a_{n-1} < a_n$ and $a_n > 0$ we obtain $a_{n+1} - a_n = 2a_n + (a_n - a_{n-1}) \Rightarrow a_{n+1} - a_n > a_n - a_{n-1} \Rightarrow a_{n+1} - a_n > 0 \Leftrightarrow a_{n+1} > a_n$ and, therefore, $a_{n+1} > 0.$ Thus, by Math Induction $a_n > 0, \forall n \in \mathbb{N}.$